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Infinite Matrices and Almost Convergence

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Abstract. In the present paper, we characterize $(r^q(u,p) : f_\infty)$, $(r^q(u,p) : f)$ and $(r^q(u,p) : f_0)$; where f_∞, f and f_0 denotes, respectively, the spaces of almost bounded sequences, almost convergent sequences and almost sequences converging to zero, where the space $r^q(u,p)$ of non-absolute type have recently been introduced by Neyaz and Hamid (see, [13]).

1. Introduction, Background and Notation

Let ω denote the space of all sequences(real or complex). The family under pointwise addition and scalar multiplication forms a linear(vector)space over real of complex numbers. Any subspace of ω is called the sequence space. So the sequence space is the set of scalar sequences(real of complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively.

Let X and Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the A-transformation from X *into* Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $A \in (X : Y)$ we mean the characterizations of matrices from X to Y *i.e.*, $A : X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \tag{1}$$

Let l_∞ and c be Banach spaces of bounded and convergent sequences $x = \{x_n\}_{n=0}^{\infty}$ with supremum norm $\|x\| = \sup_n |x_n|$. Let T denote the shift operator on ω, that is, $Tx = \{x_n\}_{n=1}^{\infty}$, $T^2x = \{x_n\}_{n=2}^{\infty}$ and so on. A Banach limit L is defined on l_∞ as a non-negative linear functional such that L is invariant *i.e.*, $L(Sx) = L(x)$ and

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$L(e) = 1,\ e = (1, 1, 1, ...)$ (see, [2]).

Lorentz (see, [7]), called a sequence $\{x_n\}$ almost convergent if all Banach limits of x, $L(x)$, are same and this unique Banach limit is called F-limit of x. In his paper, Lorentz proved the following criterion for almost convergent sequences.

A sequence $x = \{x_n\} \in l_\infty$ is almost convergent with F-limit $L(x)$ if and only if

$$\lim_{m\to\infty} t_{mn}(x) = L(x)$$

where, $\qquad t_{mn}(x) = \frac{1}{m}\sum_{j=0}^{m-1} T^j x_n,\ (T^0 = 0)$ uniformly in $n \geq 0$.

We denote the set of almost convergent sequences by f.

Nanda (see, [11]) has defined a new set of sequences f_∞ as follows:

$$f_\infty = \left\{ x \in l_\infty : \sup_{mn} |t_{mn}(x)| < \infty \right\}.$$

We call f_∞ as the set of all almost bounded sequences.

A infinite matrix $A = (a_{nk})$ is said to be regular (see, [11]) if and only if the following conditions (or Toeplitz conditions) hold:

$$(i)\quad \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1,$$

$$(ii)\ \lim_{n\to\infty} a_{nk} = 0,\ \ (k = 0, 1, 2, ...),$$

$$(iii)\ \sum_{k=0}^{\infty} |a_{nk}| < M,\ \ (M > 0,\ j = 0, 1, 2, ...).$$

Let (q_k) be a sequence of positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_k$ for $n \in \mathbb{N}$.
Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n) is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ (see, [1, 11]).

Following [1-3, 6-8, 12-14], the sequence space $r^q(u, p)$ is defined as

$$r^q(u,p) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} \right\} < \infty\ ,\ (0 < p_k \leq H = \sup p_k < \infty)$$

where, $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$.

With the notation of (1), we may re-define $r^q(u,p)$ as follows:

$$r^q(u,p) = \{l(p)\}_{R_u^q}.$$

Define the sequence $y = (y_k)$, which will be used, by the R_u^q-transform of a sequence $x = (x_k)$, *i.e.*,

$$y_k = \frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j x_j. \tag{2}$$

We denote by X^β the $\beta-$ dual of X and mean the set of all sequences $x = (x_k)$ such that $xy = (x_k y_k) \in cs$ for all $y = (y_k) \in X$.

2. Main Results

We first state some lemmas which are needed in proving the main results.

Lemma 2.1 [13]. Define the sets $D_1(u,p)$ and $D_2(u,p)$ as follows,

$$D_1(u,p) = \left\{ a = (a_k) \in \omega : \sup_k \left| \triangle\left(\frac{a_k}{u_k q_k}\right) Q_k \right|^{p_k} < \infty \text{ and } \sup_k \left| \frac{a_k}{u_k q_k} Q_k \right|^{p_k} < \infty \right\}$$

and

$$D_2(u,p) = \left\{ a = (a_k) \in \omega : \sum_k \left| \triangle\left(\frac{a_k}{u_k q_k}\right) Q_k C^{-1} \right|^{p'_k} < \infty \text{ and } \left\{ \left(\frac{a_k}{u_k q_k} Q_k C^{-1}\right)^{p'_k} \right\} \in l_\infty \right\}.$$

Then,

$[r^q(u,p)]^\beta = D_1(u,p);\ (0 < p_k \leq 1)$ and $[r^q(u,p)]^\beta = D_2(u,p); (1 < p_k \leq H < \infty)$.

Lemma 2.2 [13]. $r^q(u,p)$ is a complete linear metric space paranormed by g where

$$g(x) = \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} \right]^{\frac{1}{M}} \text{ with } 0 < p_k \leq H < \infty,\ M = \max\{1, H\}.$$

Lemma 2.3 [13]. The Riesz sequence space $r^q(u,p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq H < \infty$.

Lemma 2.4 [7]. $f \subset f_\infty$.

For simplicity in notation, we shall write

$$t_{mn}(Ax) = \frac{1}{m+1} \sum_{j=0}^{m} A_{n+j}(x) = \sum_k a(n,k,m) x_k$$

where,

$$a(n,k,m) = \frac{1}{m+1}\sum_{j=0}^{m} a_{n+j,k};\ (n,k,m \in \mathbb{N}).$$

Also,

$$\widehat{a}(n,k,m) = \triangle\left[\frac{a(n,k,m)}{u_k q_k}\right] Q_k$$

where,

$$\triangle\left[\frac{a(n,k,m)}{u_k q_k}\right] Q_k = \left[\frac{a(n,k,m)}{u_k q_k} - \frac{a(n,k+1,m)}{u_{k+1} q_{k+1}}\right] Q_k.$$

Theorem 2.5.(i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p) : f_\infty)$ if and only if there exists an integer $C > 1$ such that

$$\sup_{n,m\in\mathbb{N}} \sum_k \left|\widehat{a}(n,k,m)C^{-1}\right|^{p'_k} < \infty, \tag{3}$$

$$\left\{\left(\frac{a_{nk}}{u_k q_k} Q_k C^{-1}\right)^{p'_k}\right\} \in l_\infty \ for\ all\ n \in \mathbb{N}. \tag{4}$$

(ii) Let $0 \leq p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p) : f_\infty)$ if and only if

$$\sup_{n,k,m\in\mathbb{N}} \left|\widehat{a}(n,k,m)\right|^{p_k} < \infty. \tag{5}$$

Proof. Since $f_\infty = l_\infty$, so the proof follows from [Th. 3.1, 13]. □

Theorem 2.6. Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p) : f)$ if and only if (3), (4), (5) holds of Theorem 2.5 and there is a sequence of scalars (β_k) such that

$$\lim_{m\to\infty} \widehat{a}(n,k,m) = \beta_k,\ \text{uniformly in } n \in \mathbb{N}. \tag{6}$$

Proof: Sufficiency. Suppose that the conditions (3), (4), (5) and (6) hold and $x \in r^q(u,p)$. Then Ax exists and we have by (6) that $|a(n,k,m)C^{-1}|^{p'_k} \to \left|\beta_k C^{-1}\right|^{p'_k}$ as $m \to \infty$, uniformly in n and for each $k \in \mathbb{N}$, which leads us with (3) to the inequality

$$\sum_{j=0}^{k} \left|\beta_j C^{-1}\right|^{p'_j} = \lim_{m\to\infty} \sum_{j=0}^{k} \left|\widehat{a}(n,j,m)C^{-1}\right|^{p'_k}\ \text{(uniformly in } n)$$

$$\leq \sup_{n,m\in\mathbb{N}} \sum_{j=0}^{k} \left|\widehat{a}(n,j,m)C^{-1}\right|^{p'_k} < \infty, \tag{7}$$

holding for every $k \in \mathbb{N}$. Since $x \in r^q(u,p)$ by hypothesis and $r^q(u,p) \cong lp$ (by Lemma 2.3 above), we have $y \in l(p)$. Therefore, it follows from (7) that the series $\sum_k \beta_k y_k$ and $\sum_k \widehat{a}(n,k,m)y_k$ converges for each m,n and $y \in l(p)$. Now, for a given $\epsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$ such that $\left(\sum_{k=k_0+1}^{\infty} |y_k|^{p_k}\right)^{\frac{1}{p_k}} < \epsilon$. Then, there is some $m_0 \in \mathbb{N}$ such that $\left|\sum_{k=0}^{k_0} [\widehat{a}(n,k,m) - \beta_k]\right| < \epsilon$ for every $m \geq m_0$, uniformly in n. Since (6) holds, it follows that $\left|\sum_{k_0+1}^{\infty} [\widehat{a}(n,k,m) - \beta_k]\right|$ is arbitrary small. Therefore, $\lim_m \sum_k a(n,k,m)x_k = \lim_m \sum_k \widehat{a}(n,k,m)y_k = \sum_k \beta_k y_k$ uniformly in n. Hence, $Ax \in f$, this proves sufficiency.

Necessity. Suppose that $A \in (r^q(u,p) : f)$. Then, since $f \subset f_\infty$ (by Lemma 2.4 above), the necessities of (3) and (4) are immediately obtained from Theorem 2.3 (above). To prove the necessity of (6), consider the sequence

$$b_n^k(q) = \begin{cases} (-1)^{n-k} \frac{Q_k}{u_n q_n}, & \text{if } k \leq n \leq k+1, \\ \\ 0, & \text{if } 0 \leq n < k \text{ or } n > k+1. \end{cases}$$

Since Ax exists and is in f for each $x \in r^q(u,p)$, one can easily see that $Ab^{(k)}(q) = \left\{\triangle\left(\frac{a_{nk}}{u_k q_k}\right) Q_k\right\}_{n\in\mathbb{N}} \in f$ for all $k \in \mathbb{N}$, which proves the necessity of (6). This concludes the proof. □

Theorem 2.7. Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p) : f_0)$ if and only if (3), (4), (5) and (6) holds with $\beta_k \to 0$ for each $k \in \mathbb{N}$.

Proof. The proof follows from above Theorem 2.6 by taking $\beta_k \to 0$ for each $k \in \mathbb{N}$ □.

Corollary 2.8. By taking $u_k = 1$ for all $k \in \mathbb{N}$, we get results obtained by Jalal and Ganie [5].

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